

Non-Abelian Tensor Gauge Fields

George Savvidy
Demokritos National Research Center
Institute of Nuclear Physics
Ag. Paraskevi, GR-15310 Athens, Greece

Abstract

Recently proposed extension of Yang-Mills theory contains non-Abelian tensor gauge fields. The Lagrangian has quadratic kinetic terms, as well as cubic and quartic terms describing non-linear interaction of tensor gauge fields with the dimensionless coupling constant. We analyze particle content of non-Abelian tensor gauge fields. In four-dimensional spacetime the rank-2 gauge field describes propagating modes of helicity 2 and 0. We introduce interaction of the non-Abelian tensor gauge field with fermions and demonstrate that the free equation of motion for the spin-vector field correctly describes the propagation of massless modes of helicity 3/2. We have found a new metric-independent gauge invariant density which is a four-dimensional analog of the Chern-Simons density. The Lagrangian augmented by this Chern-Simons-like invariant describes massive Yang-Mills boson, providing a gauge-invariant mass gap for a four-dimensional gauge field theory.

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1 Introduction

It is appealing to extend the Yang-Mills theory [1, 2] so that it will define the interaction of fields which carry not only non-commutative internal charges, but also arbitrary large spins. This extension will induce the interaction of matter fields mediated by charged gauge quanta carrying spin larger than one [3]. In our recent approach these gauge fields are defined as rank- $(s + 1)$ tensors [3, 4, 5, 18]

$$A_{\mu\lambda_1\dots\lambda_s}^a(x)$$

and are totally symmetric with respect to the indices $\lambda_1\dots\lambda_s$. The index s runs from zero to infinity. The first member of this family of the tensor gauge bosons is the Yang-Mills vector boson A_μ^a .

The extended non-Abelian gauge transformation δ_ξ of the tensor gauge fields comprises a closed algebraic structure [3, 4, 5]. This allows to define generalized field strength tensors $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$, which are *transforming homogeneously* with respect to the extended gauge transformations δ_ξ . The field strength tensors are used to construct two infinite series of gauge invariant quadratic forms \mathcal{L}_s and \mathcal{L}'_s . These forms contain quadratic kinetic terms and terms describing nonlinear interaction of Yang-Mills type. In order to make all tensor gauge fields dynamical one should add all these forms in the Lagrangian [3, 4, 5].

The fermions are defined as Rarita-Schwinger spinor-tensors [20, 21, 22]

$$\psi_{\lambda_1\dots\lambda_s}^\alpha(x)$$

with mixed transformation properties of Dirac four-component wave function (the index α denotes the Dirac index) and are totally symmetric tensors of the rank s over the indices $\lambda_1\dots\lambda_s$. All fields of the $\{\psi\}$ family are isotopic multiplets belonging to the same representation σ of the compact Lie group G (the corresponding indices are suppressed). The gauge invariant Lagrangian for fermions also contains a linear sum of two infinite series of forms $\mathcal{L}_{s+1/2}$ and $\mathcal{L}'_{s+1/2}$. The coupling constants in front of these forms remain arbitrary because all terms in the sum are separately gauge invariant. The extended gauge symmetry alone does not define them. *The basic principle which we shall pursue in our construction will be to fix these coupling constants demanding unitarity of the theory*¹.

In the second and third sections we shall outline the transformation properties of non-Abelian tensor gauge fields, the definition of the corresponding field stress tensors, the general expression for the invariant Lagrangian and the description of propagating modes for the lower rank tensor gauge fields [3, 4, 5, 23]. In the forth and fifth sections we shall incorporate into the theory fermions of half-integer spins [19]. We shall construct two infinite series of gauge invariant forms and study the propagating modes for lower rank fermion fields.

In the sixth sections we shall construct a metric-independent gauge invariant density which is a four-dimensional analog of the Chern-Simons density. The Lagrangian augmented by this Chern-Simons-like invariant describes massive Yang-Mills boson, providing a gauge-invariant mass gap for a four-dimensional gauge field theory [53].

¹For that one should study the spectrum of the theory and to prove that there are no propagating negative norm states, that is, ghost states.

2 Non-Abelian Tensor Gauge Fields

The gauge fields are defined as rank-\$(s + 1)\$ tensors [3]

$$A_{\mu\lambda_1\ldots\lambda_s}^a(x), \quad s = 0, 1, 2, \dots$$

and are totally symmetric with respect to the indices \$\lambda_1\ldots\lambda_s\$. A priori the tensor fields have no symmetries with respect to the first index \$\mu\$. The index \$a\$ numerates the generators \$L^a\$ of the Lie algebra \$\mathfrak{g}\$ of a *compact*² Lie group \$G\$. One can think of these tensor fields as appearing in the expansion of the extended gauge field \$\mathcal{A}_\mu(x, e)\$ over the unite vector \$e_\lambda\$ [5]:

$$\mathcal{A}_\mu(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} A_{\mu\lambda_1\ldots\lambda_s}^a(x) L^a e_{\lambda_1}\ldots e_{\lambda_s}. \quad (1)$$

The gauge field \$A_{\mu\lambda_1\ldots\lambda_s}^a\$ carries indices \$a, \lambda_1, \dots, \lambda_s\$ labeling the generators of *extended current algebra* \$\mathcal{G}\$ associated with compact Lie group \$G\$. It has infinitely many generators \$L_{\lambda_1\ldots\lambda_s}^a = L^a e_{\lambda_1}\ldots e_{\lambda_s}\$ and the corresponding algebra is given by the commutator [5]

$$[L_{\lambda_1\ldots\lambda_i}^a, L_{\lambda_{i+1}\ldots\lambda_s}^b] = if^{abc} L_{\lambda_1\ldots\lambda_s}^c. \quad (2)$$

Because \$L_{\lambda_1\ldots\lambda_s}^a\$ are space-time tensors, the full algebra includes the Poincaré generators \$P_\mu, M_{\mu\nu}\$. They act on the space-time components of the above generators as follows [19]:

$$\begin{aligned} [P^\mu, P^\nu] &= 0, \\ [M^{\mu\nu}, P^\lambda] &= i(\eta^{\lambda\nu} P^\mu - \eta^{\lambda\mu} P^\nu), \\ [M^{\mu\nu}, M^{\lambda\rho}] &= i(\eta^{\mu\rho} M^{\nu\lambda} - \eta^{\mu\lambda} M^{\nu\rho} + \eta^{\nu\lambda} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\lambda}), \\ [P^\mu, L_a^{\lambda_1\ldots\lambda_s}] &= 0, \\ [M^{\mu\nu}, L_a^{\lambda_1\ldots\lambda_s}] &= i(\eta^{\lambda_1\nu} L_a^{\mu\lambda_2\ldots\lambda_s} - \eta^{\lambda_1\mu} L_a^{\nu\lambda_2\ldots\lambda_s} + \dots + \eta^{\lambda_s\nu} L_a^{\lambda_1\ldots\lambda_{s-1}\mu} - \eta^{\lambda_s\mu} L_a^{\lambda_1\ldots\lambda_{s-1}\nu}), \\ [L_a^{\lambda_1\ldots\lambda_i}, L_b^{\lambda_{i+1}\ldots\lambda_s}] &= if_{abc} L_c^{\lambda_1\ldots\lambda_s}, \quad (\mu, \nu, \rho, \lambda = 0, 1, 2, 3; \quad s = 0, 1, 2, \dots) \end{aligned} \quad (3)$$

It is an infinite-dimensional extension of the Poincaré algebra by generators which contains isospin algebra \$G\$. In some sense the new vector variable \$e_\lambda\$ plays a role similar to the Grassmann variable \$\theta\$ in supersymmetry algebras [10, 11].

The extended non-Abelian gauge transformations of the tensor gauge fields are defined by the following equations [4]:

$$\begin{aligned} \delta A_\mu^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi^b, \\ \delta A_{\mu\nu}^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi_\nu^b + g f^{acb} A_{\mu\nu}^c \xi^b, \\ \delta A_{\mu\nu\lambda}^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi_{\nu\lambda}^b + g f^{acb} (A_{\mu\nu}^c \xi_\lambda^b + A_{\mu\lambda}^c \xi_\nu^b + A_{\nu\lambda}^c \xi_\mu^b), \\ \dots\dots\dots &\dots\dots\dots \end{aligned} \quad (4)$$

where \$\xi_{\lambda_1\ldots\lambda_s}^a(x)\$ are totally symmetric gauge parameters. These extended gauge transformations generate a closed algebraic structure. In order to see that one should compute the commutator of two extended gauge transformations \$\delta_\eta\$ and \$\delta_\xi\$ of parameters \$\eta\$ and \$\xi\$. The commutator of two transformations can be expressed in the following form [4]:

$$[\delta_\eta, \delta_\xi] A_{\mu\lambda_1\lambda_2\ldots\lambda_s} = -ig \delta_\zeta A_{\mu\lambda_1\lambda_2\ldots\lambda_s} \quad (5)$$

²The algebra \$\mathfrak{g}\$ possesses an orthogonal basis in which the structure constants \$f^{abc}\$ are totally antisymmetric.

and is again an extended gauge transformation with the gauge parameters $\{\zeta\}$ which are given by the matrix commutators

$$\begin{aligned}\zeta &= [\eta, \xi] \\ \zeta_{\lambda_1} &= [\eta, \xi_{\lambda_1}] + [\eta_{\lambda_1}, \xi] \\ \zeta_{\nu\lambda} &= [\eta, \xi_{\nu\lambda}] + [\eta_\nu, \xi_\lambda] + [\eta_\lambda, \xi_\nu] + [\eta_{\nu\lambda}, \xi], \\ \dots &\dots\end{aligned}\tag{6}$$

Each single field $A_{\mu\lambda_1\dots\lambda_s}^a(x)$, $s = 1, 2, 3, \dots$ has no geometrical interpretation, but their union has a geometrical interpretation in terms of *connection on the extended vector bundle* X [5]. Indeed, one can define the extended vector bundle X whose structure group is \mathcal{G} with group elements $U(\xi) = \exp[i\xi(x, e)]$, where

$$\xi(x, e) = \sum_s \frac{1}{s!} \xi_{\lambda_1\dots\lambda_s}^a(x) L^a e_{\lambda_1}\dots e_{\lambda_s}.$$

Defining the extended gauge transformation of $\mathcal{A}_\mu(x, e)$ in a standard way

$$\mathcal{A}'_\mu(x, e) = U(\xi)\mathcal{A}_\mu(x, e)U^{-1}(\xi) - \frac{i}{g}\partial_\mu U(\xi) U^{-1}(\xi),\tag{7}$$

we get the extended vector bundle X on which the gauge field $\mathcal{A}_\mu^a(x, e)$ is a connection [2]. The expansion of (7) over the vector e_λ reproduces gauge transformation law of the tensor gauge fields (4). Using the commutator of the covariant derivatives $\nabla_\mu^{ab} = (\partial_\mu - ig\mathcal{A}_\mu(x, e))^{ab}$

$$[\nabla_\mu, \nabla_\nu]^{ab} = gf^{abc}\mathcal{G}_{\mu\nu}^c,\tag{8}$$

we can define the extended field strength tensor

$$\mathcal{G}_{\mu\nu}(x, e) = \partial_\mu\mathcal{A}_\nu(x, e) - \partial_\nu\mathcal{A}_\mu(x, e) - ig[\mathcal{A}_\mu(x, e)\mathcal{A}_\nu(x, e)]\tag{9}$$

which transforms homogeneously: $\mathcal{G}'_{\mu\nu}(x, e) = U(\xi)\mathcal{G}_{\mu\nu}(x, e)U^{-1}(\xi)$. Thus the *generalized field strengths are defined as* [4]

$$\begin{aligned}G_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c, \\ G_{\mu\nu,\lambda}^a &= \partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a + gf^{abc}(A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c), \\ G_{\mu\nu,\lambda\rho}^a &= \partial_\mu A_{\nu\lambda\rho}^a - \partial_\nu A_{\mu\lambda\rho}^a + gf^{abc}(A_\mu^b A_{\nu\lambda\rho}^c + A_{\mu\lambda}^b A_{\nu\rho}^c + A_{\mu\rho}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho}^b A_\nu^c), \\ \dots &\dots\end{aligned}\tag{10}$$

and transform homogeneously with respect to the extended gauge transformations (4). The field strength tensors are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices. The inhomogeneous extended gauge transformation (4) induces the homogeneous gauge transformation of the corresponding field strength tensors (10) of the form [4]

$$\begin{aligned}\delta G_{\mu\nu}^a &= gf^{abc}G_{\mu\nu}^b \xi^c, \\ \delta G_{\mu\nu,\lambda}^a &= gf^{abc}(G_{\mu\nu,\lambda}^b \xi^c + G_{\mu\nu}^b \xi_\lambda^c), \\ \delta G_{\mu\nu,\lambda\rho}^a &= gf^{abc}(G_{\mu\nu,\lambda\rho}^b \xi^c + G_{\mu\nu,\lambda}^b \xi_\rho^c + G_{\mu\nu,\rho}^b \xi_\lambda^c + G_{\mu\nu}^b \xi_{\lambda\rho}^c), \\ \dots &\dots\end{aligned}\tag{11}$$

Using these field strength tensors one can construct two infinite series of forms \mathcal{L}_s and \mathcal{L}'_s ($s = 2, 3, \dots$) invariant with respect to the transformations δ_ξ . They are quadratic in field strength tensors. The first series is given by the formula [3, 4, 5]

$$\begin{aligned}\mathcal{L}_{s+1} &= -\frac{1}{4} G_{\mu\nu,\lambda_1\dots\lambda_s}^a G_{\mu\nu,\lambda_1\dots\lambda_s}^a + \dots\dots \\ &= -\frac{1}{4} \sum_{i=0}^{2s} a_i^s G_{\mu\nu,\lambda_1\dots\lambda_i}^a G_{\mu\nu,\lambda_{i+1}\dots\lambda_{2s}}^a \left(\sum_P \eta^{\lambda_{i1}\lambda_{i2}} \dots \eta^{\lambda_{i2s-1}\lambda_{i2s}} \right),\end{aligned}\quad (12)$$

where the sum \sum_P runs over all nonequal permutations of λ_i 's, in total $(2s-1)!!$ terms, and the numerical coefficients are $a_i^s = \frac{s!}{i!(2s-i)!}$. The second series of gauge invariant quadratic forms is given by the formula [3, 4, 5]

$$\begin{aligned}\mathcal{L}'_{s+1} &= \frac{1}{4} G_{\mu\nu,\rho\lambda_3\dots\lambda_{s+1}}^a G_{\mu\rho,\nu\lambda_3\dots\lambda_{s+1}}^a + \frac{1}{4} G_{\mu\nu,\nu\lambda_3\dots\lambda_{s+1}}^a G_{\mu\rho,\rho\lambda_3\dots\lambda_{s+1}}^a + \dots\dots \\ &= \frac{1}{4} \sum_{i=1}^{2s+1} \frac{a_{i-1}^s}{s} G_{\mu\lambda_1,\lambda_2\dots\lambda_i}^a G_{\mu\lambda_{i+1},\lambda_{i+2}\dots\lambda_{2s+2}}^a \left(\sum_P' \eta^{\lambda_{i1}\lambda_{i2}} \dots \eta^{\lambda_{i2s+1}\lambda_{i2s+2}} \right),\end{aligned}\quad (13)$$

where the sum \sum_P' runs over all nonequal permutations of λ_i 's, with exclusion of the terms which contain $\eta^{\lambda_1,\lambda_{i+1}}$.

These forms contain quadratic *kinetic terms*, as well as cubic and quartic terms describing *nonlinear interaction of gauge fields* with dimensionless coupling constant g . In order to make all tensor gauge fields dynamical one should add all these forms in the Lagrangian [3, 4, 5]:

$$\mathcal{L} = \mathcal{L}_{YM} + (\mathcal{L}_2 + \mathcal{L}'_2) + g_3(\mathcal{L}_3 + \frac{4}{3}\mathcal{L}'_3) + \dots + g_{s+1}(\mathcal{L}_{s+1} + \frac{2s}{s+1}\mathcal{L}'_{s+1}) + \dots\quad (14)$$

The coupling constants g_3, g_4, \dots remain arbitrary because each term is separately invariant with respect to the extended gauge transformations δ_ξ and leaves these coupling constants yet undetermined.

In the next section we shall analyze the free field equations for the lower rank non-Abelian tensor gauge fields [23]. These equations are written in terms of the first order derivatives of extended field strength tensors, similarly to the electrodynamics and the Yang-Mills theory. In the Yang-Mills theory the free equation of motion describes the propagation of massless gauge bosons of helicity $\lambda = \pm 1$. The rank-2 gauge field describes propagating modes of helicity two and zero: $\lambda = \pm 2, 0$. Thus the lower rank gauge fields have the following helicity content of propagating modes:

$$A_\mu : \quad \lambda = \pm 1, \quad A_{\mu\nu} : \quad \lambda = \pm 2, 0.\quad (15)$$

The propagating modes of higher rank gauge fields have been analyzed in [23].

3 Propagating Modes of Tensor Gauge Bosons

In the Yang-Mills theory the free field equation is defined by the quadratic form:

$$\mathcal{L}_{YM}|_{quadratic} = \frac{1}{2} A_\alpha^a \mathcal{H}_{\alpha\gamma} A_\gamma^a,$$

where

$$\mathcal{H}_{\alpha\gamma} = -k^2\eta_{\alpha\gamma} + k_\alpha k_\gamma. \quad (16)$$

and describes the propagation of the massless gauge bosons of helicity $\lambda = \pm 1$:

$$e_\mu^\pm = (0, 1, \pm i, 0).$$

The kinetic term of the rank-2 gauge field is given by the quadratic form:

$$\mathcal{L}_2 + \mathcal{L}'_2|_{quadratic} = \frac{1}{2} A_{\alpha\dot{\alpha}}^a \mathcal{H}_{\alpha\dot{\alpha}\gamma\dot{\gamma}} A_{\gamma\dot{\gamma}}^a, \quad (17)$$

where the kinetic operator is [3, 4, 5]

$$\begin{aligned} \mathcal{H}_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) = & (-\eta_{\alpha\gamma}\eta_{\dot{\alpha}\dot{\gamma}} + \frac{1}{2}\eta_{\alpha\dot{\gamma}}\eta_{\dot{\alpha}\gamma} + \frac{1}{2}\eta_{\alpha\dot{\alpha}}\eta_{\gamma\dot{\gamma}})k^2 + \eta_{\alpha\gamma}k_{\dot{\alpha}}k_{\dot{\gamma}} + \eta_{\dot{\alpha}\dot{\gamma}}k_\alpha k_\gamma \\ & - \frac{1}{2}(\eta_{\alpha\dot{\gamma}}k_{\dot{\alpha}}k_\gamma + \eta_{\dot{\alpha}\gamma}k_\alpha k_{\dot{\gamma}} + \eta_{\alpha\dot{\alpha}}k_\gamma k_{\dot{\gamma}} + \eta_{\gamma\dot{\gamma}}k_\alpha k_{\dot{\alpha}}). \end{aligned} \quad (18)$$

Thus one should solve the free equation of motion:

$$\mathcal{H}_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) e_{\gamma\dot{\gamma}}(k) = 0. \quad (19)$$

The vector space of independent solutions $A_{\gamma\dot{\gamma}} = e_{\gamma\dot{\gamma}}(k)e^{ikx}$ depends on the rank of the matrix $\mathcal{H}_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k)$. Because the matrix operator $\mathcal{H}_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k)$ explicitly depends on the momentum k_μ , its $rank \mathcal{H} = r(k)$ also depends on momenta and therefore the number of independent solutions \mathcal{N} depends on momenta $\mathcal{N}(k) = d - r(k)$. The $rank \mathcal{H}$ is a Lorentz invariant quantity and therefore depends on the value of momentum square k_μ^2 .

The matrix operator (18) in the four-dimensional space-time is a 16×16 matrix³. In the reference frame, where $k^\gamma = (\omega, 0, 0, k)$, it has a particularly simple form. If $\omega^2 - k^2 \neq 0$, the rank of the 16-dimensional matrix $H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k)$ is equal to $rank H|_{\omega^2 - k^2 \neq 0} = 9$ and the number of linearly independent solutions is $16 - 9 = 7$. These seven solutions are pure gauge fields

$$e_{\gamma\dot{\gamma}} = k_\gamma \xi_{\dot{\gamma}} + k_{\dot{\gamma}} \zeta_\gamma, \quad (20)$$

where ξ_γ and ζ_γ are independent gauge parameters. When $\omega^2 - k^2 = 0$, then the rank of the matrix $H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k)$ drops and $rank H|_{\omega^2 - k^2 = 0} = 6$. This leaves us with $16 - 6 = 10$ solutions. These are 7 solutions, the pure gauge potentials (20), and three new solutions representing the propagating modes:

$$e_{\gamma\dot{\gamma}}^\pm = e_\gamma^\pm e_{\dot{\gamma}}^\pm, \quad e_{\gamma\dot{\gamma}}^A = e_\gamma^+ e_{\dot{\gamma}}^- - e_\gamma^- e_{\dot{\gamma}}^+. \quad (21)$$

Thus the general solution of the equation on the mass-shell is

$$e_{\gamma\dot{\gamma}} = \xi_\gamma k_{\dot{\gamma}} + \zeta_\gamma k_\gamma + c_1 e_{\gamma\dot{\gamma}}^+ + c_2 e_{\gamma\dot{\gamma}}^- + c_3 e_{\gamma\dot{\gamma}}^A, \quad (22)$$

where c_1, c_2, c_3 are arbitrary constants. These are the propagating modes of *helicity-two* and *helicity-zero* $\lambda = \pm 2, 0$ charged gauge bosons [3, 4, 5]. The propagating modes of higher rank gauge fields have been found in [23].

³The multi-index $N \equiv (\mu, \lambda)$ takes sixteen values.

4 Invariant Forms for Fermions

The fermions are defined as Rarita-Schwinger spinor-tensor fields [20, 21, 22]

$$\psi_{\lambda_1 \dots \lambda_s}^\alpha(x) \quad (23)$$

with mixed transformation properties of Dirac four-component wave function and are totally symmetric tensors of the rank s over the indices $\lambda_1 \dots \lambda_s$ (the index α denotes the Dirac index and will be suppressed in the rest part of the article). All fields of the $\{\psi\}$ family are isotopic multiplets $\psi_{\lambda_1 \dots \lambda_s}^i(x)$ belonging to the same representation σ_{ij}^a of the compact Lie group G (the index i denotes the isotopic index). One can think of these spinor-tensor fields as appearing in the expansion of the extended fermion field $\Psi^i(x, e)$ over the unit tangent vector e_λ [3, 5]:

$$\Psi^i(x, e) = \sum_{s=0}^{\infty} \psi_{\lambda_1 \dots \lambda_s}^i(x) e_{\lambda_1} \dots e_{\lambda_s}. \quad (24)$$

Our intention is to introduce gauge invariant interaction of fermion fields with non-Abelian tensor gauge fields. The transformation of the fermions under the extended isotopic group we shall define by the formula [4]

$$\Psi'(x, e) = \mathcal{U}(\xi) \Psi(x, e), \quad (25)$$

where $\mathcal{U}(\xi) = \exp(ig\xi(x, e))$, $\xi(x, e) = \sum_{s=0}^{\infty} \xi_{\lambda_1 \dots \lambda_s}^a(x) \sigma^a e_{\lambda_1} \dots e_{\lambda_s}$ and σ^a are the matrices of the representation σ of the compact Lie group G , according to which all ψ' s are transforming. In components the transformation of fermion fields under the extended isotopic group therefore will be [4]

$$\begin{aligned} \delta_\xi \psi &= ig \sigma^a \xi^a \psi, \\ \delta_\xi \psi_\lambda &= ig \sigma^a (\xi^a \psi_\lambda + \xi_\lambda^a \psi), \\ \delta_\xi \psi_{\lambda\rho} &= ig \sigma^a (\xi^a \psi_{\lambda\rho} + \xi_\lambda^a \psi_\rho + \xi_\rho^a \psi_\lambda + \xi_{\lambda\rho}^a \psi), \\ \dots\dots\dots &\dots\dots\dots \end{aligned} \quad (26)$$

The covariant derivative of the fermion field is defined as usually:

$$\nabla_\mu \Psi = i\partial_\mu \Psi + g\mathcal{A}_\mu(x, e)\Psi, \quad (27)$$

and transforms homogeneously: $\nabla_\mu \Psi \rightarrow \mathcal{U} \nabla_\mu \Psi$, where we are using the matrix notation for the gauge fields $\mathcal{A}_\mu = \sigma^a \mathcal{A}_\mu^a$. Therefore the gauge invariant Lagrangian has the following form:

$$\mathcal{L}^F = \bar{\Psi} \gamma_\mu [i\partial_\mu \Psi + g\mathcal{A}_\mu] \Psi. \quad (28)$$

Expanding this Lagrangian over the vector variable e_λ one can get a series of gauge invariant forms for half-integer fermion fields:

$$\mathcal{L}^F = \sum_{s=0}^{\infty} \mathcal{L}_{s+1/2}, \quad (29)$$

where f_s are coupling constants. The lower-spin invariant Lagrangian for the spin-1/2 field is:

$$\mathcal{L}_{1/2} = \bar{\psi}^i \gamma_\mu (\delta_{ij} i\partial_\mu + g\sigma_{ij}^a A_\mu^a) \psi^j = \bar{\psi} (i \not{\partial} + g \not{A}) \psi \quad (30)$$

and for the spin-vector field ψ_μ together with the additional rank-2 spin-tensor $\psi_{\mu\nu}$ the invariant Lagrangian has the form [4]:

$$\begin{aligned}\mathcal{L}_{3/2} = & \bar{\psi}_\lambda \gamma_\mu (i\partial_\mu + gA_\mu) \psi_\lambda + \frac{1}{2} \bar{\psi} \gamma_\mu (i\partial_\mu + gA_\mu) \psi_{\lambda\lambda} + \frac{1}{2} \bar{\psi}_{\lambda\lambda} \gamma_\mu (i\partial_\mu + gA_\mu) \psi \\ & + g\bar{\psi}_\lambda \gamma_\mu A_{\mu\lambda} \psi + g\bar{\psi} \gamma_\mu A_{\mu\lambda} \psi_\lambda + \frac{1}{2} g\bar{\psi} \gamma_\mu A_{\mu\lambda\lambda} \psi.\end{aligned}\quad (31)$$

As one can check it is invariant under simultaneous gauge transformations of the fermions (26) and tensor gauge fields (4):

$$\delta\mathcal{L}_{3/2} = 0.$$

The Lagrangian (29) is not the most general Lagrangian which can be constructed in terms of the above spinor-tensor fields (23). As we shall see, there exists a second invariant \mathcal{L}'_F which can be constructed in terms of spinor-tensor fields (23), and the total Lagrangian is a linear sum: $\mathcal{L}_F + f \mathcal{L}'_F$.

Let us consider the gauge invariant tensor density of the form [3, 5]

$$\mathcal{L}_{\rho_1\rho_2} = \bar{\Psi}(x, e) \gamma_{\rho_1} [i\partial_{\rho_2} + g\sigma^a \mathcal{A}_{\rho_2}^a(x, e)] \Psi(x, e). \quad (32)$$

It is gauge invariant tensor density because its variation is equal to zero:

$$\begin{aligned}\delta\mathcal{L}_{\rho_1\rho_2}(x, e) = & i\bar{\Psi}(x, e) \xi(x, e) \gamma_{\rho_1} [i\partial_{\rho_2} + g\mathcal{A}_{\rho_2}(x, e)] \Psi(x, e) + \\ & + \bar{\Psi}(x, e) \gamma_{\rho_1} g(-\frac{1}{g}) [\partial_{\rho_2} \xi(x, e) - ig[\mathcal{A}_{\rho_2}(x, e), \xi(x, e)] \Psi(x, e) + \\ & - i\bar{\Psi}(x, e) \gamma_{\rho_1} [i\partial_{\rho_2} + g\sigma^a \mathcal{A}_{\rho_2}^a(x, e)] \xi(x, e) \Psi(x, e) = 0,\end{aligned}$$

where $\mathcal{A}_{\rho_2}(x, e) = \sigma^a \mathcal{A}_{\rho_2}^a(x, e)$. The Lagrangian density (32) generates the series of *gauge invariant tensor densities* $(\mathcal{L}'_{\rho_1\rho_2})_{\lambda_1\dots\lambda_s}(x)$, when we expand it in powers of the vector variable e :

$$\mathcal{L}_{\rho_1\rho_2}(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} (\mathcal{L}_{\rho_1\rho_2})_{\lambda_1\dots\lambda_s}(x) e_{\lambda_1} \dots e_{\lambda_s}. \quad (33)$$

The gauge invariant tensor densities $(\mathcal{L}_{\rho_1\rho_2})_{\lambda_1\dots\lambda_s}(x)$ allow to construct two series of gauge invariant forms: $\mathcal{L}_{s+1/2}$ and $\mathcal{L}'_{s+1/2}$, $s=1,2,\dots$ by the contraction of the corresponding tensor indices. The lower gauge invariant tensor density has the form

$$\begin{aligned}(\mathcal{L}_{\rho_1\rho_2})_{\lambda_1\lambda_2} = & \frac{1}{2} \{ + \bar{\psi}_{\lambda_1} \gamma_{\rho_1} [i\partial_{\rho_2} + gA_{\rho_2}] \psi_{\lambda_2} + \bar{\psi}_{\lambda_2} \gamma_{\rho_1} [i\partial_{\rho_2} + gA_{\rho_2}] \psi_{\lambda_1} + \\ & + \bar{\psi}_{\lambda_1\lambda_2} \gamma_{\rho_1} [i\partial_{\rho_2} + gA_{\rho_2}] \psi + \bar{\psi} \gamma_{\rho_1} [i\partial_{\rho_2} + gA_{\rho_2}] \psi_{\lambda_1\lambda_2} + \\ & + g\bar{\psi}_{\lambda_1} \gamma_{\rho_1} A_{\rho_2\lambda_2} \psi + g\bar{\psi}_{\lambda_2} \gamma_{\rho_1} A_{\rho_2\lambda_1} \psi + \\ & + g\bar{\psi} \gamma_{\rho_1} A_{\rho_2\lambda_2} \psi_{\lambda_1} + g\bar{\psi} \gamma_{\rho_1} A_{\rho_2\lambda_1} \psi_{\lambda_2} + g\bar{\psi} \gamma_{\rho_1} A_{\rho_2\lambda_1\lambda_2} \psi \},\end{aligned}\quad (34)$$

and we shall use it to generate Lorentz invariant densities. Performing contraction of the indices of this tensor density with respect to $\eta_{\rho_1\rho_2} \eta_{\lambda_1\lambda_2}$ we shall reproduce our first gauge invariant Lagrangian density $\mathcal{L}_{3/2}$ (31) presented in the previous section. We shall get the second gauge invariant Lagrangian performing the contraction with respect to the $\eta_{\rho_1\lambda_1} \eta_{\rho_2\lambda_2}$, which is obviously different from the previous one:

$$\begin{aligned}\mathcal{L}'_{3/2} = & \frac{1}{2} \{ \bar{\psi}_\mu \gamma_\mu (i\partial_\lambda + gA_\lambda) \psi_\lambda + \bar{\psi}_\lambda (i\partial_\lambda + gA_\lambda) \gamma_\mu \psi_\mu + \\ & + \bar{\psi}_{\mu\lambda} \gamma_\mu (i\partial_\lambda + gA_\lambda) \psi + \bar{\psi} (i\partial_\lambda + gA_\lambda) \gamma_\mu \psi_{\mu\lambda} + \\ & + g\bar{\psi}_\mu \gamma_\lambda A_{\mu\lambda} \psi + g\bar{\psi} \gamma_\mu A_{\lambda\mu} \psi_\lambda + g\bar{\psi}_\mu \gamma_\mu A_{\lambda\lambda} \psi + g\bar{\psi} \gamma_\mu A_{\lambda\lambda} \psi_\mu + g\bar{\psi} \gamma_\mu A_{\lambda\mu\lambda} \psi \}.\end{aligned}\quad (35)$$

Independently, one can check that the last expression is invariant under simultaneous extended gauge transformations of fermions (26) and tensor gauge fields (4), calculating its variation:

$$\delta \mathcal{L}'_{3/2} = 0.$$

As one can see from (31) and (35) the interaction of fermions with tensor gauge bosons is going through the cubic vertex which includes two fermions and a tensor gauge boson, very similar to the vertices in QED and the Yang-Mills theory.

Thus the gauge invariant Lagrangian for fermions contains two infinite series of invariant forms $\mathcal{L}_{s+1/2}$, $\mathcal{L}'_{s+1/2}$ and is a linear sum of these forms

$$\mathcal{L}^F = \mathcal{L}_{1/2} + \sum_{s=1}^{\infty} (\mathcal{L}_{s+1/2} + f_s \mathcal{L}'_{s+1/2}). \quad (36)$$

It is important to notice that the invariance with respect to the extended gauge transformations does not fix the coupling constants f_s . The coupling constants f_s remain arbitrary because every term of the sum is separately gauge invariant and the extended gauge symmetry alone does not define them. *The basic principle which we shall pursue in our construction will be to fix these coupling constants demanding unitarity of the theory*⁴.

5 Propagating Modes of Tensor Fermions

As we have seen above the Lagrangian for lower rank fields is a linear sum

$$\mathcal{L} = \mathcal{L}_{1/2} + \mathcal{L}_{3/2} + f_1 \mathcal{L}'_{3/2} + \dots$$

and our aim is to find out the coefficient f_1 for which the equation for the spin-vector field ψ_λ correctly describes the propagation of spin 3/2. In the limit $g \rightarrow 0$ the kinetic part of the Lagrangian will take the form

$$\begin{aligned} \mathcal{L}_{3/2} + f_1 \mathcal{L}'_{3/2}|_{kinetic} &= \bar{\psi}_\lambda \gamma_\mu i \partial_\mu \psi_\lambda + \frac{1}{2} \bar{\psi} \gamma_\mu i \partial_\mu \psi_{\lambda\lambda} + \frac{1}{2} \bar{\psi}_{\lambda\lambda} \gamma_\mu i \partial_\mu \psi \\ &+ \frac{f_1}{2} \{ \bar{\psi}_\mu \gamma_\mu i \partial_\lambda \psi_\lambda + \bar{\psi}_\lambda i \partial_\lambda \gamma_\mu \psi_\mu + \bar{\psi}_{\mu\lambda} \gamma_\mu i \partial_\lambda \psi + \bar{\psi} i \partial_\lambda \gamma_\mu \psi_{\mu\lambda} \} \end{aligned}$$

and we have the following free equation of motion for the spin-vector field ψ_λ :

$$\not{p} \psi_\mu + f_1 \frac{1}{2} (\gamma_\mu p_\lambda + \gamma_\lambda p_\mu) \psi_\lambda = 0,$$

where $\not{p} = \gamma_\mu p_\mu = \gamma_\mu i \partial_\mu$. Let us consider the gauge transformation of the spin-vector field ψ_λ of the Rarita-Schwinger form:

$$\delta \psi_\lambda = \partial_\lambda \varepsilon. \quad (37)$$

The variation of the equation shows that for $f_1 = -2$ the equation is invariant with respect to the RS transformation, if the spinor parameter ε fulfils the condition

$$\partial^2 \varepsilon = 0. \quad (38)$$

⁴For that one should study the spectrum of the theory and its dependence on these coupling constants. For some particular values of coupling constants the linear sum of these forms may exhibit symmetries with respect to a bigger gauge group securing the absence of ghost states.

The equation for the spin-vector field ψ_λ therefore is

$$H_{\mu\lambda}u_\lambda = (\not{p}\eta_{\mu\lambda} - \gamma_\mu p_\lambda - \gamma_\lambda p_\mu)u_\lambda = 0, \quad (39)$$

where the matrix operator is $H_{\mu\lambda} = \eta_{\mu\lambda} \not{p} - \gamma_\mu p_\lambda - \gamma_\lambda p_\mu$ and $\psi_\lambda = u_\lambda \exp(ipx)$.

Let us first consider the Dirac equation for the spinor $\psi = u \exp(ipx)$ and the Lagrangian $\mathcal{L}_{1/2}$ (30):

$$\not{p}u = 0.$$

It describes particles of helicities $\lambda = \pm 1/2$, so that the wave functions are

$$\omega = +p, \quad u_+ = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_- = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega = -p, \quad v_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where we have chosen the momentum vector in the third direction $p^\mu = (\omega, 0, 0, p)$.

Now let us consider the matrix operator $H_{\mu\lambda}$ for spin-vector ψ_λ . If $\omega^2 - p^2 \neq 0$, the rank of the 16-dimensional matrix $H_{\mu\lambda}$ is $\text{rank } H|_{\omega^2 - p^2 \neq 0} = 16$ and we have only trivial solution $u_\lambda = 0$. If $\omega = +p$, then the rank of the matrix drops, $\text{rank } H|_{\omega = +p} = 10$, and the number of independent solutions is $16 - 10 = 6$. These six solutions of the equation (39) are

$$u_\lambda^\alpha \text{ (gauge)} = p_\lambda \otimes \varepsilon^\alpha, \quad u_\lambda^\alpha = e_\mu^\pm \otimes u_\pm^\alpha \quad (40)$$

where ε^α is a spinor gauge parameter. The first four solutions are pure gauge fields ($\sim p_\lambda$), while the remaining two are the physical modes of helicities $\lambda = \pm 3/2$. If $\omega = -p$ we have again gauge modes and two physical modes of helicities $\lambda = \pm 3/2$ describing antiparticles. The general solution on the mass-shell will be a linear combination of all these solutions. We conclude that equation (39) correctly describes the free propagation of physical modes of the massless particle of spin $3/2$.

In summary we have the following massless spectrum for the lower rank boson and fermion fields:

$$\begin{array}{ll} A_\mu : & \lambda = \pm 1 \\ A_{\mu\nu} : & \lambda = \pm 2, 0 \end{array} \quad \begin{array}{ll} \psi : & \lambda = \pm 1/2, \\ \psi_\mu : & \lambda = \pm 3/2, \end{array} \quad (41)$$

.....

The propagating modes of higher rank gauge fields have been analyzed in [23].

6 Topological Mass Generation

Several mechanisms are currently known for generating massive vector particles that are compatible with the gauge invariance. One of them is the spontaneous symmetry breaking mechanism, which generates masses and requires the existence of the fundamental scalar particle - the Higgs boson. The scalar field provides the longitudinal polarization of the massive vector boson and ensures unitarity of its scattering amplitudes [24, 25]⁵.

⁵Extended discussion and references can be found in [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

The argument in favor of a pure gauge field theory mechanism was a dynamical mechanism of mass generation proposed by Schwinger [26], who was arguing that the gauge invariance of a vector field does not necessarily lead to the massless spectrum of its excitations and suggested its realization in (1+1)-dimensional gauge theory[27].

Compatibility of gauge invariance and mass term in (2+1)-dimensional gauge field theory was demonstrated by Deser, Jackiw and Templeton [38, 39] and Schonfeld [40], who added to the YM Lagrangian a gauge invariant Chern-Simons density.

Here we suggest a similar mechanism that generates masses of the YM boson and tensor gauge bosons in (3+1)-dimensional space-time at the classical level [53]. As we shall see, in non-Abelian tensor gauge theory [3, 4, 5] there exists a gauge invariant, metric-independent density Γ in five-dimensional space-time which is the derivative of the vector current Σ_μ . This invariant in five dimensions has many properties of the Chern-Pontryagin density $\mathcal{P} = \partial_\mu C_\mu$ in four-dimensional YM theory, which is a derivative of the Chern-Simons topological vector current C_μ . The fifth component of the vector current $\Sigma_4 \equiv \Sigma$ is a gauge invariant density which is defined in four-dimensional space-time. Its dimensionality is $[mass]^3$. Adding this term to the Lagrangian of non-Abelian tensor gauge fields leaves intact its gauge invariance and generates massive vector particles of the mass

$$M^2 = \frac{4}{3} m^2. \quad (42)$$

A massive spin-1 particle appears here as a vector field of helicities $\lambda = \pm 1$ which acquires a third longitudinal polarization absorbing antisymmetric field which carries a zero helicity $\lambda = 0$ (15).

Let us consider this new invariant in five-dimensional space-time (4 + 1), which can be constructed by means of the totally antisymmetric Levi-Civita epsilon tensor $\varepsilon_{\mu\nu\lambda\rho\sigma}$ ($\mu, \nu, \dots = 0, 1, 2, 3, 4$) in combination with the generalized field strength tensors (10) [53]

$$\Gamma = \varepsilon_{\mu\nu\lambda\rho\sigma} Tr G_{\mu\nu} G_{\lambda\rho,\sigma} = 2 \varepsilon_{\mu\nu\lambda\rho\sigma} G_{\mu\nu}^a G_{\lambda\rho,\sigma}^a. \quad (43)$$

We shall demonstrate that this invariant in five dimensions has many properties of the Chern-Pontryagin density

$$\mathcal{P} = \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho} Tr G_{\mu\nu} G_{\lambda\rho} = \partial_\mu C_\mu \quad (44)$$

in Yang-Mill theory in four dimensions, where

$$C_\mu = \varepsilon_{\mu\nu\lambda\rho} Tr (A_\nu \partial_\lambda A_\rho - i \frac{2}{3} g A_\nu A_\lambda A_\rho) \quad (45)$$

is the Chern-Simons topological current. Indeed, Γ is obviously diffeomorphism-invariant and does not involve a space-time metric. It is gauge invariant because under the gauge transformation δ_ξ (4) it vanishes:

$$\delta_\xi \Gamma = -ig \varepsilon_{\mu\nu\lambda\rho\sigma} Tr ([G_{\mu\nu} \xi] G_{\lambda\rho,\sigma} + G_{\mu\nu} ([G_{\lambda\rho,\sigma} \xi] + [G_{\lambda\rho} \xi_\sigma])) = 0.$$

The variation of its integral over the gauge fields A_μ^a and $A_{\mu\lambda}^a$ gives:

$$\begin{aligned} \delta_A \int_{M_5} d^5 x \Gamma &= -2 \varepsilon_{\mu\nu\lambda\rho\sigma} \int d^5 x Tr ((\nabla_\mu G_{\lambda\rho,\sigma} - ig [A_{\mu\sigma} G_{\lambda\rho}]) \delta A_\nu + (\nabla_\lambda G_{\mu\nu}) \delta A_{\rho\sigma}) \\ &+ 2 \varepsilon_{\mu\nu\lambda\rho\sigma} \int d^5 x Tr (\nabla_\mu (G_{\lambda\rho,\sigma} \delta A_\nu) + \nabla_\lambda (G_{\mu\nu} \delta A_{\rho\sigma})). \end{aligned}$$

Recalling the Bianchi identity in the YM theory and the generalized Bianchi identities for higher-rank field strength tensor $G_{\nu\lambda,\rho}$ presented in the Appendix, one can see that Γ gets contribution only from the boundary terms and vanishes when the fields vary in the bulk of the manifold⁶:

$$\begin{aligned}\delta_A \int_{M_5} d^5x \Gamma &= 2\varepsilon_{\mu\nu\lambda\rho\sigma} \int_{M_5} d^5x \partial_\mu \text{Tr}(G_{\lambda\rho,\sigma} \delta A_\nu + G_{\nu\lambda} \delta A_{\rho\sigma}) = \\ &= 2\varepsilon_{\mu\nu\lambda\rho\sigma} \int_{\partial M_5} \text{Tr}(G_{\lambda\rho,\sigma} \delta A_\nu + G_{\nu\lambda} \delta A_{\rho\sigma}) d\sigma_\mu = 0.\end{aligned}$$

Therefore Γ is insensitive to the *local* variation of the fields. It becomes obvious that Γ is a total derivative of some vector current Σ_μ . Indeed, simple algebraic computation gives $\Gamma = \varepsilon_{\mu\nu\lambda\rho\sigma} \text{Tr} G_{\mu\nu} G_{\lambda\rho,\sigma} = \partial_\mu \Sigma_\mu$, where

$$\Sigma_\mu = 2\varepsilon_{\mu\nu\lambda\rho\sigma} \text{Tr}(A_\nu \partial_\lambda A_{\rho\sigma} - \partial_\lambda A_\nu A_{\rho\sigma} - 2ig A_\nu A_\lambda A_{\rho\sigma}). \quad (46)$$

After some rearrangement and taking into account the definition of the field strength tensors (10) we can get the following form of the vector current [53]:

$$\Sigma_\mu = \varepsilon_{\mu\nu\lambda\rho\sigma} \text{Tr} G_{\nu\lambda} A_{\rho\sigma}. \quad (47)$$

It is instructive to compare the expressions (43), (44) and (45), (47). Both entities \mathcal{P} and Γ are metric-independent, are insensitive to the local variation of the fields and are derivatives of the corresponding vector currents C_μ and Σ_μ . The difference between them is that the former is defined in four dimensions, while the latter in five. This difference in one unit of the space-time dimension originates from the fact that we have at our disposal high-rank tensor gauge fields to build new invariants. The same is true for the Chern-Simons topological current C_μ and for the current Σ_μ , where the latter is defined in five dimensions. It is also remarkable that the current Σ_μ is linear in the YM field strength tensor and in the rank-2 gauge field, picking up only its antisymmetric part.

While the invariant Γ and the vector current Σ_μ are defined on a five-dimensional manifold, we may restrict the latter to one lower, four-dimensional manifold. The restriction proceeds as follows. Let us consider the fifth component of the vector current Σ_μ :

$$\Sigma \equiv \Sigma_4 = \varepsilon_{4\nu\lambda\rho\sigma} \text{Tr} G_{\nu\lambda} A_{\rho\sigma}. \quad (48)$$

Considering the fifth component of the vector current $\Sigma \equiv \Sigma_4$ one can see that the remaining indices will not repeat the external index and the sum is restricted to the sum over indices of four-dimensional space-time. Therefore we can reduce this functional to four dimensions. This is the case when the gauge fields are independent on the fifth coordinate x_4 . Thus the density Σ is well defined in four-dimensional space-time and, as we shall see, it is also gauge invariant up to the total divergence term. Therefore we shall consider its integral over four-dimensional space-time⁷:

$$\int_{M_4} d^4x \Sigma = \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} d^4x \text{Tr} G_{\nu\lambda} A_{\rho\sigma}. \quad (49)$$

⁶The trace of the commutators vanishes: $\text{Tr}([A_\mu; G_{\lambda\rho,\sigma} \delta A_\nu] + [A_\lambda; G_{\mu\nu} \delta A_{\rho\sigma}]) = 0$.

⁷Below we are using the same Greek letters to numerate now the four-dimensional coordinates. There should be no confusion because the dimension can always be recovered from the dimension of the epsilon tensor.

This entity is an analog of the Chern-Simon secondary characteristic

$$CS = \varepsilon_{ijk} \int_{M_3} d^3x \operatorname{Tr} (A_i \partial_j A_k - ig \frac{2}{3} A_i A_j A_k), \quad (50)$$

but, importantly, instead of being defined in three dimensions it is now defined in four dimensions. Thus the non-Abelian tensor gauge fields allow to build a natural generalization of the Chern-Simons characteristic in four-dimensional space-time.

As we claimed this functional is gauge invariant up to the total divergence term. Indeed, its gauge variation under δ_ξ (4) is

$$\begin{aligned} \delta_\xi \int_{M_4} d^4x \Sigma &= \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} \operatorname{Tr} (-ig[G_{\nu\lambda} \xi] A_{\rho\sigma} + G_{\nu\lambda} (\nabla_\rho \xi_\sigma - ig[A_{\rho\sigma} \xi])) d^4x = \\ &= \varepsilon_{\nu\lambda\rho\sigma} \int_{M_4} \partial_\rho \operatorname{Tr} (G_{\nu\lambda} \xi_\sigma) d^4x = \varepsilon_{\nu\lambda\rho\sigma} \int_{\partial M_4} \operatorname{Tr} (G_{\nu\lambda} \xi_\sigma) d\sigma_\rho = 0. \end{aligned} \quad (51)$$

Here the first and the third terms cancel each other and the second one, after integration by part and recalling the Bianchi identity (59), leaves only the boundary term, which vanishes when the gauge parameter ξ_σ tends to zero at infinity.

It is interesting to know whether the invariant Σ is associated with some new topological characteristic of the gauge fields. If the YM field strength $G_{\nu\lambda}$ vanishes, then the vector potential is equal to the pure gauge connection $A_\mu = U^{-1} \partial_\mu U$. Inspecting the expression for the invariant Σ one can get convinced that it vanishes on such fields because there is a field strength tensor $G_{\nu\lambda}$ in the integrant. Therefore it does not differentiate topological properties of the gauge function U , like its winding number. Both "small" and "large" gauge transformations have zero contribution to this invariant. It may distinguish fields which are falling less faster at infinity and have nonzero field strength tensor $G_{\nu\lambda}$ and the tensor gauge field $A_{\rho\sigma}$.

In four dimensions the gauge fields have dimension of $[mass]^1$, therefore if we intend to add this new density to the Lagrangian we should introduce the mass parameter m :

$$m \Sigma = m \varepsilon_{\nu\lambda\rho\sigma} \operatorname{Tr} G_{\nu\lambda} A_{\rho\sigma}, \quad (52)$$

where parameter m has units $[mass]^1$. Adding this term to the Lagrangian of non-Abelian tensor gauge fields keeps intact its gauge invariance and our aim is to analyze the particle spectrum of this gauge field theory. The natural appearance of the mass parameters hints at the fact that the theory turns out to be a massive theory. We shall see that the YM vector boson becomes massive, suggesting an alternative mechanism for mass generation in gauge field theories in four-dimensional space-time.

We have to notice that the Abelian version of the invariant Σ was investigated earlier in [41, 42, 43, 44, 45, 46, 47, 48, 49]. Indeed, if one considers instead of a non-Abelian group the Abelian group one can see that the invariant Σ reduces to the $\varepsilon_{\nu\lambda\rho\sigma} F_{\nu\lambda} B_{\rho\sigma}$ and when added to the Maxwell Lagrangian provides a mass to the vector field [41, 43, 45, 44, 46, 47]. Attempts at producing a non-Abelian invariant in a similar way have come up with difficulties because they involve non-Abelian generalization of gauge transformations of antisymmetric fields [47, 48, 50, 51, 52]. Let us compare the formulas (2.16) and (2.17) suggested in [47, 48] for the transformation of antisymmetric field with the gauge transformation δ_ξ (4). For lower-rank fields the latter can be written in the following way:

$$\begin{aligned} \delta_\xi A_\mu &= \partial_\mu \xi - ig[A_\mu, \xi], & \delta_\xi A_{\mu\nu} &= -ig[A_{\mu\nu}, \xi], \\ \delta_\zeta A_\mu &= 0, & \delta_\zeta A_{\mu\nu} &= \partial_\mu \zeta_\nu - ig[A_\mu, \zeta_\nu]. \end{aligned}$$

The antisymmetric part of this transformation amazingly coincides with the one suggested in [47] if one takes the auxiliary field A_μ^i of [47] equal to zero. The crucial point is that the gauge transformations of non-Abelian tensor gauge fields [3, 4, 5] defined in (4) cannot be limited to a YM vector and antisymmetric field $B_{\mu\nu}^a$. Instead, antisymmetric field is augmented by a symmetric rank-2 gauge field, so that together they form a gauge field $A_{\mu\nu}^a$ which transforms as it is given above and is a fully propagating field. It is also important that one should include all high-rank gauge fields in order to be able to close the group of gauge transformations and to construct invariant Lagrangian.

Let us now see how the spectrum is changing when we add new invariant Σ (52) to the Lagrangian. With the new mass term the Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_2 + \mathcal{L}'_2 + \frac{m}{4} \Sigma. \quad (53)$$

The free equations ($g=0$) of motion for the YM and rank-2 gauge fields are:

$$\begin{aligned} \partial^2 A_\nu^a - \partial_\nu \partial_\mu A_\mu^a + m \varepsilon_{\nu\mu\lambda\rho} \partial_\mu A_{\lambda\rho}^a &= 0, \\ \partial^2 (A_{\nu\lambda}^a - \frac{1}{2} A_{\lambda\nu}^a) - \partial_\nu \partial_\mu (A_{\mu\lambda}^a - \frac{1}{2} A_{\lambda\mu}^a) - \partial_\lambda \partial_\mu (A_{\nu\mu}^a - \frac{1}{2} A_{\mu\nu}^a) + \\ + \partial_\nu \partial_\lambda (A_{\mu\mu}^a - \frac{1}{2} A_{\mu\mu}^a) + \frac{1}{2} \eta_{\nu\lambda} (\partial_\mu \partial_\rho A_{\mu\rho}^a - \partial^2 A_{\mu\mu}^a) + m \varepsilon_{\nu\lambda\mu\rho} \partial_\mu A_\rho^a &= 0. \end{aligned} \quad (54)$$

This is a coupled system of equations which involve the vector YM field and the antisymmetric part of the rank-2 gauge field. Only the antisymmetric part $B_{\nu\lambda}$ of the rank-2 gauge field $A_{\nu\lambda}$ interacts through the mass term, the symmetric part $A_{\nu\lambda}^S$ completely decouples from both equations⁸, therefore we arrive at the following system of equations:

$$\begin{aligned} (-k^2 \eta_{\nu\mu} + k_\nu k_\mu) e_\mu + i m \varepsilon_{\nu\mu\lambda\rho} k_\mu b_{\lambda\rho} &= 0, \\ (-k^2 \eta_{\nu\mu} \eta_{\lambda\rho} + k_\nu k_\mu \eta_{\lambda\rho} - \eta_{\nu\mu} k_\lambda k_\mu) b_{\mu\rho} + i \frac{2m}{3} \varepsilon_{\nu\lambda\mu\rho} k_\mu e_\rho &= 0. \end{aligned} \quad (55)$$

When $k^2 \neq M^2$ the system (55) is off mass-shell and we have four pure gauge field solutions:

$$e_\mu = k_\mu, \quad b_{\nu\lambda} = 0; \quad e_\mu = 0, \quad b_{\nu\lambda} = k_\nu \xi_\lambda - k_\lambda \xi_\nu. \quad (56)$$

When $k^2 = M^2$ the system (55) has seven solutions. These are four pure gauge solutions (56) and additional three solutions representing propagating modes:

$$\begin{aligned} e_\mu^{(1)} &= (0, 1, 0, 0), \quad b_{\gamma\dot{\gamma}}^{(1)} = \frac{1}{i} \frac{M}{\sqrt{k^2 + M^2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\ e_\mu^{(2)} &= (0, 0, 1, 0), \quad b_{\gamma\dot{\gamma}}^{(2)} = -\frac{1}{i} \frac{M}{\sqrt{k^2 + M^2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ e_\mu^{(3)} &= (0, 0, 0, \frac{M}{\sqrt{k^2 + M^2}}), \quad b_{\gamma\dot{\gamma}}^{(3)} = \frac{1}{i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (57)$$

⁸The symmetric field can acquire a mass when we include the next invariant mass term $m_3 \Sigma_3$ [53].

These propagating modes cannot be factorized into separately vector or separately tensor solutions as it happens for the pure gauge solutions (56). It is a genuine superposition of vector and tensor fields. Let us consider the limit $M \rightarrow 0$. The above solutions will factorize into two massless vector modes $e_\mu^{(1)}$, $e_\mu^{(2)}$, of helicities $\lambda = \pm 1$ and helicity $\lambda = 0$ mode $b_{\gamma\gamma}^{(3)}$ of antisymmetric tensor. But when $M \neq 0$, in the rest frame $\vec{k}^2 = 0$, these solutions represent three polarizations of the spin-1 boson.

The above analysis suggests the following physical interpretation. A massive spin-1 particle appears here as a vector field of helicities $\lambda = \pm 1$ which acquires an extra polarization state absorbing antisymmetric field of helicity $\lambda = 0$, or as antisymmetric field of helicity $\lambda = 0$ which absorbs helicities $\lambda = \pm 1$ of the vector field. It is sort of "dual" description of massive spin-1 particle. In order to fully justify this phenomenon of superposition of polarizations one should develop quantum-mechanical description of tensor fields. There is a need for deeper understanding of the corresponding path integral which is over infinitely many fields.

In conclusion, I wish to thank the organizers of the conference for the invitation and for arranging an interesting and stimulating meeting. I also would like to thank Irina Aref'eva, Ludwig Faddeev and Andrey Slavnov for interesting discussions during the conference.

7 Appendix

The field strength tensors fulfil the Bianchi identities. In YM theory it is

$$[\nabla_\mu, G_{\nu\lambda}] + [\nabla_\nu, G_{\lambda\mu}] + [\nabla_\lambda, G_{\mu\nu}] = 0, \quad (58)$$

for the higher rank field strength tensors $G_{\nu\lambda,\rho}$ and $G_{\nu\lambda,\rho\sigma}$ they are:

$$[\nabla_\mu, G_{\nu\lambda,\rho}] - ig[A_{\mu\rho}, G_{\nu\lambda}] + [\nabla_\nu, G_{\lambda\mu,\rho}] - ig[A_{\nu\rho}, G_{\lambda\mu}] + [\nabla_\lambda, G_{\mu\nu,\rho}] - ig[A_{\lambda\rho}, G_{\mu\nu}] = 0, \quad (59)$$

$$[\nabla_\mu, G_{\nu\lambda,\rho\sigma}] - ig[A_{\mu\rho}, G_{\nu\lambda,\sigma}] - ig[A_{\mu\sigma}, G_{\nu\lambda,\rho}] - ig[A_{\mu\rho\sigma}, G_{\nu\lambda}] + cyc.perm.(\mu\nu\lambda) = 0 \quad (60)$$

and so on.

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